Journal of Number Theory 131 (2011) 508–518

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Cyclic sum of certain parametrized multiple series

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ARTICLE INFO

Article history:
Received 6 March 2009
Revised 15 February 2010
Accepted 7 September 2010
Available online 24 November 2010
Communicated by D. Zagier

Keywords:
Cyclic sum
Parametrized multiple series
Multiple zeta value
Multiple zeta-star value
Hurwitz zeta-function

ABSTRACT

In the present paper, we prove the cyclic sum formulas for certain parametrized multiple series.

1. Introduction

The multiple zeta value (MZV for short) is defined by the multiple series

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_n) := \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $k_1, \dots, k_n \in \mathbb{Z}_{\geqslant 1}$ and $k_n \geqslant 2$. The case $n = 2$ was studied by L. Euler in [10], and the general case was introduced by M.E. Hoffman [14] and D. Zagier [36]. As an object similar to MZV, the multiple zeta-star value (MZSV for short) is defined by the multiple series

$$\zeta^\star(\mathbf{k}) = \zeta^\star(k_1, \dots, k_n) := \sum_{0<m_1\leqslant\cdots\leqslant m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.09.011

(see [10,14]). MZVs and MZSVs satisfy various relations (see, e.g., [14,15,18,20,24,30] and [1,2,22,29, 31], respectively), though the formulations are different in some relations (cf., e.g., [15] and [29]).

In [16,17], the author studied the parametrized multiple series

$$Z(\mathbf{k};\alpha)=Z(k_1,\ldots,k_n;\alpha):=\sum_{0\leqslant m_1<\cdots<m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n}}\frac{1}{(m_1+\alpha)^{k_1}\cdots(m_n+\alpha)^{k_n}},$$

where $k_1,\ldots,k_n\in\mathbb{Z}_{\geqslant 1}$, $k_n\geqslant 2$, $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$, and $(a)_m$ denotes the Pochhammer symbol defined by

$$(a)_m=\begin{cases}a(a+1)\cdots(a+m-1) & \text{if } m\in\mathbb{Z}_{\geqslant 1},\\ 1 & \text{if } m=0.\end{cases}$$

Obviously we can consider the following analogue of $\zeta^\star(\mathbf{k})$:

$$Z^\star(\mathbf{k};\alpha)=Z^\star(k_1,\ldots,k_n;\alpha):=\sum_{0\leqslant m_1\leqslant\cdots\leqslant m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n}}\frac{1}{(m_1+\alpha)^{k_1}\cdots(m_n+\alpha)^{k_n}}.$$

By the above definitions, we immediately see that $Z(\mathbf{k};1)=\zeta(\mathbf{k})$, $Z^\star(\mathbf{k};1)=\zeta^\star(\mathbf{k})$ and $Z(k_1;\alpha)=Z^\star(k_1;\alpha)=\zeta(k_1;\alpha)$, where $\zeta(s;\alpha):=\sum_{m=0}^{\infty}(m+\alpha)^{-s}$ is the Hurwitz zeta-function; by this fact, the multiple series $Z(\mathbf{k};\alpha)$ and $Z^\star(\mathbf{k};\alpha)$ can be regarded as multiple versions of the Hurwitz zeta-function. In [16], the author found $Z(\mathbf{k};\alpha)$ and a slightly general multiple series (see also [17, Introduction]) by studying Ochiai's proof of the sum formula for MZVs ([23]; see also Remark 2.10 below). We note that a multiple series like $Z^\star(\mathbf{k};\alpha)$ was studied by M. Émery in [9]. (However, as was stated in [9], the results in the first version of [9] follow from some results of others published before [9] (see, e.g., [11, Proposition 2.1]).) We also note that Krattenthaler and Rivoal's hypergeometric identity [21, Proposition 1 (ii)] contains a relation among $Z^\star(\mathbf{k};\alpha)$ (for the details, see Remark 2.7 below). In [17], the author proved that the multiple series $Z(\mathbf{k};\alpha)$ satisfy the same relation as Ohno's relation for MZVs [24, Theorem 1]. This result means that the multiple series $Z(\mathbf{k};\alpha)$ satisfy many $\mathbb{Q}$-linear relations. Further the multiple series $Z(\mathbf{k};\alpha)$ have some good properties for us to study relations among them. For example, by differentiating both sides of the duality formula for $Z(\mathbf{k};\alpha)$ (see [17, Lemma 2.4]), we can get a relation among $Z(\mathbf{k};\alpha)$ which has the same form as Hoffman's relation for MZVs [14, Theorem 5.1]. For another example, see [17, Theorem 1.2]. These results follow from the fact that the derivative of $Z(\mathbf{k};\alpha)$ can be expressed as a certain $\mathbb{Z}$-linear combination of $Z(\mathbf{k};\alpha)$. For $Z^\star(\mathbf{k};\alpha)$, it is easy to verify that

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}\alpha}Z^\star(k_1,\ldots,k_n;\alpha)&=\sum_{i=1}^{n}(k_i-1+\delta_{1i})Z^\star(k_1,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots,k_n;\alpha)\\&\quad+\sum_{i=1}^{n-1}Z^\star(k_1,\ldots,k_i,1,k_{i+1},\ldots,k_n;\alpha),\end{aligned}$$

where δ_{ij} denotes Kronecker's delta. This is also useful for the study of relations among $Z^\star(\mathbf{k};\alpha)$. Further we remark that relations among $Z(\mathbf{k};\alpha)$ and $Z^\star(\mathbf{k};\alpha)$ can be applied to the study of relations among MZVs and MZSVs (see [17] and Remark 2.6 below). By the above facts, we think that to find relations among $Z(\mathbf{k};\alpha)$ and $Z^\star(\mathbf{k};\alpha)$ is interesting.

In the present paper, we prove that the multiple series $Z(\mathbf{k};\alpha)$ and $Z^\star(\mathbf{k};\alpha)$ satisfy the same relations as the cyclic sum formulas for MZVs (M.E. Hoffman and Y. Ohno [15]; see also [25]) and MZSVs (Y. Ohno and N. Wakabayashi [29, Theorem 1]), respectively. Namely we shall prove the following:

Theorem 1.1. *For any positive integers $k_1, \ldots, k_n$ with $k_i \geqslant 2$ for some i and all complex numbers α with positive real part, the following identities hold*:

(i)

$$\sum_{i=1}^{n} \sum_{j=0}^{k_i-2} Z(j+1, k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i - j; \alpha) = \sum_{i=1}^{n} Z(k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i + 1; \alpha);$$

(ii)

$$\sum_{i=1}^{n} \sum_{j=0}^{k_i-2} Z^{\star}(j+1, k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i - j; \alpha) = k\zeta(k+1; \alpha),$$

where $k := k_1 + \cdots + k_n$.

In the above identities, the empty sums are interpreted as 0.

Theorem 1.1 is a generalization of the cyclic sum formulas for MZVs and MZSVs. Indeed, taking $\alpha = 1$ in Theorem 1.1 (i) and (ii), we get the cyclic sum formulas for MZVs and MZSVs, respectively. Theorem 1.1 also yields the sum formulas for $Z(\mathbf{k}; \alpha)$ and $Z^{\star}(\mathbf{k}; \alpha)$ (see Corollary 2.8 below).

In [34], T. Tanaka and N. Wakabayashi proved certain relations among MZVs and MZSVs which generalize the cyclic sum formulas for MZVs and MZSVs, respectively [34, Theorem 2.1, Corollary 3.4]. Some q-analogues of the cyclic sum formulas for MZVs and MZSVs can be found in [7,27,28]. These q-analogues are also the cyclic sum formulas for parametrized multiple series.

2. Proof of Theorem 1.1

In the present section, we prove Theorem 1.1. In order to prove Theorem 1.1, we use the same methods as used by M.E. Hoffman and Y. Ohno for MZVs in [15] (see also [25]: most of the contents of [25] can be also found in [26, Section 2.3]), and used by Y. Ohno and N. Wakabayashi for MZSVs in [29]. We note that T. Tanaka and N. Wakabayashi [34] gave an alternative proof of the cyclic sum formulas for MZVs and MZSVs by using Kawashima's relations for MZVs and MZSVs [20, Corollary 5.5, Remark 5.6]. (We note that, in [20, p. 756], G. Kawashima conjectured that his relation among MZVs contained the cyclic sum formula for MZVs.)

Now we consider the multiple series

$$T(k_1, \ldots, k_n; \alpha) := \sum_{0 \leqslant m_0 < m_1 < \cdots < m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_1+\alpha)^{k_1} \cdots (m_n+\alpha)^{k_n} (m_n - m_0)}$$

and

$$T^{\star}(k_1, \ldots, k_n; \alpha) := \sum_{\substack{0 \leqslant m_0 \leqslant m_1 \leqslant \cdots \leqslant m_n \\ m_0 \neq m_n}} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_1+\alpha)^{k_1} \cdots (m_n+\alpha)^{k_n} (m_n - m_0)}.$$

We first prove a lemma.

Lemma 2.1. *Let $k_1, \ldots, k_n$ be positive integers and let $k_i \geqslant 2$ for some i. Then the multiple series $T(k_1, \ldots, k_n; \alpha)$ and $T^{\star}(k_1, \ldots, k_n; \alpha)$ converge absolutely for $\alpha \in \{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha > 0\}$ and uniformly in any compact subset of $\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha > 0\}$.*

Proof. We fix any real number r with $0 < r < 1$. For a given positive integer m and all $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geqslant r$, by using Stirling's formula for the gamma function, we get

$$\sum_{0 \leqslant m_0 < \cdots < m_{n-1} < m} \left| \frac{(\alpha)_{m_0}}{m_0!} \frac{m!}{(\alpha)_m} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)^{k_i}} \right\} \frac{1}{(m+\alpha)^{k_n}(m - m_0)} \right|$$

$$\leqslant \sum_{0 \leqslant m_0 < \cdots < m_{n-1} < m} \frac{(r)_{m_0}}{m_0!} \frac{m!}{(r)_m} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + r)^{k_i}} \right\} \frac{1}{(m+r)^{k_n}(m - m_0)}$$

$$\ll \frac{1}{m^{k_n + r - 1}} \sum_{0 \leqslant m_0 < \cdots < m_{n-1} < m} \frac{1}{m_1^{k_1} \cdots m_{n-1}^{k_{n-1}} (m - m_0)},$$

where the implied constant depends only on r. Further we can prove the following:

$$\sum_{m=1}^{\infty} \frac{1}{m^{k_n + r - 1}} \sum_{0 \leqslant m_0 < \cdots < m_{n-1} < m} \frac{1}{m_1^{k_1} \cdots m_{n-1}^{k_{n-1}} (m - m_0)} \leqslant \zeta(1 + r/2)\zeta(\underbrace{1, \ldots, 1}_{n-1}, 1 + r/2) < \infty$$

for $r > 0$ and $k_1, \ldots, k_n \geqslant 1$ with $k_i \geqslant 2$ for some i. Therefore, by the Weierstrass M-test, we get the assertion for $T(k_1, \ldots, k_n; \alpha)$.

By the same argument as above, we can prove the assertion for $T^\star(k_1, \ldots, k_n; \alpha)$. $\square$

We shall use the following lemma.

Lemma 2.2. *The identity*

$$\frac{(\alpha)_{m+1}}{m!} \sum_{l=n+1}^{\infty} \frac{l!}{(\alpha)_{l+1}} \frac{1}{l - m} = \frac{n!}{(\alpha)_n} \sum_{l=0}^{m-1} \frac{(\alpha)_l}{l!} \frac{1}{n - l} + \frac{(\alpha)_m}{m!} \frac{n!}{(\alpha)_{n+1}} \tag{1}$$

holds for any integers m, n with $0 \leqslant m \leqslant n$ and all complex numbers α with positive real part, where the sum on the right-hand side is interpreted as 0 if $m = 0$.

Remark 2.3. The following proof of Lemma 2.2, which is based on the idea of creating telescoping series, is due to the anonymous referee. In the proof of Lemma 2.2 in an earlier version of the present paper, we used some identities for hypergeometric series to prove a more general identity than (1). Our proof was complicated.

Proof of Lemma 2.2. The left-hand side of (1) can be written as

$$\frac{(\alpha)_m}{m!} \sum_{l=n+1}^{\infty} \frac{l!}{(\alpha)_l} \left(\frac{1}{l - m} - \frac{1}{\alpha + l} \right)$$

$$= \frac{(\alpha)_m}{m!} \sum_{l=n+1}^{\infty} \frac{l!}{(\alpha)_l} \left(\frac{1}{l - m} - \frac{1}{l} \right) + \frac{(\alpha)_m}{m!} \sum_{l=n+1}^{\infty} \frac{l!}{(\alpha)_l} \left(\frac{1}{l} - \frac{1}{\alpha + l} \right). \tag{2}$$

We note that the first and the last sums on the right-hand side of (2) converge absolutely for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. Now we put the first sum on the right-hand side of (2) as S_m. Then we can calculate S_m as follows:

$$
\begin{aligned}
S_m &= \frac{(\alpha)_m}{(m-1)!}\sum_{l=n+1}^{\infty}\frac{(l-1)!}{(\alpha)_l}\frac{1}{l-m}\\
&= \frac{(\alpha)_{m-1}}{(m-1)!}\sum_{l=n+1}^{\infty}\frac{(l-1)!}{(\alpha)_{l-1}}\left(\frac{1}{l-m}-\frac{1}{\alpha+l-1}\right)\\
&= \frac{(\alpha)_{m-1}}{(m-1)!}\frac{n!}{(\alpha)_n}\frac{1}{n+1-m}+\frac{(\alpha)_{m-1}}{(m-1)!}\sum_{l=n+1}^{\infty}\left(\frac{l!}{(\alpha)_l}\frac{1}{l+1-m}-\frac{(l-1)!}{(\alpha)_l}\right)\\
&= \frac{(\alpha)_{m-1}}{(m-1)!}\frac{n!}{(\alpha)_n}\frac{1}{n+1-m}+\frac{(\alpha)_{m-1}}{(m-1)!}\sum_{l=n+1}^{\infty}\frac{l!}{(\alpha)_l}\left(\frac{1}{l+1-m}-\frac{1}{l}\right)\\
&= \frac{(\alpha)_{m-1}}{(m-1)!}\frac{n!}{(\alpha)_n}\frac{1}{n+1-m}+S_{m-1}.
\end{aligned}
$$

Repeating this calculation m times, and noting that $S_0 = 0$, we get

$$
S_m = \frac{n!}{(\alpha)_n}\sum_{l=0}^{m-1}\frac{(\alpha)_l}{l!}\frac{1}{n-l}.
$$

For the last sum on the right-hand side of (2), we can calculate as

$$
\frac{(\alpha)_m}{m!}\sum_{l=n+1}^{\infty}\frac{l!}{(\alpha)_l}\left(\frac{1}{l}-\frac{1}{\alpha+l}\right)=\frac{(\alpha)_m}{m!}\sum_{l=n+1}^{\infty}\left(\frac{(l-1)!}{(\alpha)_l}-\frac{l!}{(\alpha)_{l+1}}\right)=\frac{(\alpha)_m}{m!}\frac{n!}{(\alpha)_{n+1}}.
$$

Therefore, combining the above identities for the first and the last sums on the right-hand side of (2), we get (1). □

Remark 2.4. For any $m, n \in \mathbb{Z}$ with $0 \leqslant m < n$, the identity (1) can be rewritten as

$$
\frac{(\alpha)_{m+1}}{m!}\sum_{l=n}^{\infty}\frac{l!}{(\alpha)_{l+1}}\frac{1}{l-m}=\frac{n!}{(\alpha)_n}\sum_{l=0}^{m}\frac{(\alpha)_l}{l!}\frac{1}{n-l}.
$$

The multiple series $T(k_1,\dots,k_n;\alpha)$ and $T^\star(k_1,\dots,k_n;\alpha)$ have the same properties as in [15, Theorem 3.2] (see also [25, Key Lemma]) and [29, Key Lemma 1], respectively.

Lemma 2.5. *For any positive integers $k_1,\dots,k_n$ with $k_i \geqslant 2$ for some i and all complex numbers α with positive real part, the following identities hold*:

(i)

$$
\begin{aligned}
&T(k_1,\dots,k_n;\alpha)-T(k_n,k_1,\dots,k_{n-1};\alpha)\\
&\quad= Z(k_n,k_1,\dots,k_{n-2},k_{n-1}+1;\alpha)-\sum_{j=0}^{k_n-2}Z(j+1,k_1,\dots,k_{n-1},k_n-j;\alpha);
\end{aligned}
$$

(ii)

$$\begin{aligned}
&T^{\star}(k_1,\ldots,k_n;\alpha)-T^{\star}(k_n,k_1,\ldots,k_{n-1};\alpha)\\
&\quad=k_n\zeta(k+1;\alpha)-\sum_{j=0}^{k_n-2}Z^{\star}(j+1,k_1,\ldots,k_{n-1},k_n-j;\alpha),
\end{aligned}$$

where $k:=k_1+\cdots+k_n$.

In the above identities, the empty sums are interpreted as 0.

Proof. In order to prove (i), we use the same method as in [15, Section 3] (see also [25, Proof of Key Lemma]). First, for any $j\in\mathbb{Z}$ with $0\leqslant j\leqslant k_n-2$, we calculate as follows:

$$\begin{aligned}
&\sum_{0\leqslant m_0<\cdots<m_{n-1}<m_n}\frac{(\alpha)_{m_0}}{m_0!}\frac{m_n!}{(\alpha)_{m_n}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^j}\frac{1}{(m_n+\alpha)^{k_n-j}(m_n-m_0)}\\
&=\sum_{0\leqslant m_0<\cdots<m_{n-1}<m_n}\frac{(\alpha)_{m_0}}{m_0!}\frac{m_n!}{(\alpha)_{m_n}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^{j+1}}\\
&\quad\times\frac{1}{(m_n+\alpha)^{k_n-j-1}}\left(\frac{1}{m_n-m_0}-\frac{1}{m_n+\alpha}\right)\\
&=\sum_{0\leqslant m_0<\cdots<m_{n-1}<m_n}\frac{(\alpha)_{m_0}}{m_0!}\frac{m_n!}{(\alpha)_{m_n}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^{j+1}}\frac{1}{(m_n+\alpha)^{k_n-j-1}(m_n-m_0)}\\
&\quad-Z(j+1,k_1,\ldots,k_{n-1},k_n-j;\alpha).
\end{aligned}$$

Summing up the above identity for all $j\in\mathbb{Z}$ with $0\leqslant j\leqslant k_n-2$, we get

$$\begin{aligned}
&T(k_1,\ldots,k_n;\alpha)\\
&\quad=\sum_{0\leqslant m_0<\cdots<m_{n-1}<m_n}\frac{(\alpha)_{m_0}}{m_0!}\frac{m_n!}{(\alpha)_{m_n}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^{k_n-1}}\frac{1}{(m_n+\alpha)(m_n-m_0)}\\
&\qquad-\sum_{j=0}^{k_n-2}Z(j+1,k_1,\ldots,k_{n-1},k_n-j;\alpha).
\end{aligned}$$

Further, by using Lemma 2.2, we can calculate the first sum on the right-hand side of the above identity as follows:

$$\begin{aligned}
&\sum_{0\leqslant m_0<\cdots<m_{n-1}<m_n}\frac{(\alpha)_{m_0}}{m_0!}\frac{m_n!}{(\alpha)_{m_n}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^{k_n-1}}\frac{1}{(m_n+\alpha)(m_n-m_0)}\\
&=\sum_{0\leqslant m_0<\cdots<m_{n-1}}\left\{\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{k_i}}\right\}\frac{1}{(m_0+\alpha)^{k_n}}\frac{(\alpha)_{m_0+1}}{m_0!}\sum_{m_n=m_{n-1}+1}^{\infty}\frac{m_n!}{(\alpha)_{m_n+1}}\frac{1}{m_n-m_0}
\end{aligned}$$

$$= \sum_{0 \leqslant m_0 < \cdots < m_{n-1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i+\alpha)^{k_i}} \right\} \frac{1}{(m_0+\alpha)^{k_n}} \left\{ \frac{m_{n-1}!}{(\alpha)_{m_{n-1}}} \sum_{l=0}^{m_0-1} \frac{(\alpha)_l}{l!} \frac{1}{m_{n-1}-l} + \frac{(\alpha)_{m_0}}{m_0!} \frac{m_{n-1}!}{(\alpha)_{m_{n-1}+1}} \right\}$$
$$= T(k_n, k_1, \ldots, k_{n-1}; \alpha) + Z(k_n, k_1, \ldots, k_{n-2}, k_{n-1}+1; \alpha).$$

This completes the proof of (i).

By the same method as in [29, Proof of Key Lemma 1] and using Lemma 2.2, we can prove (ii). □

Theorem 1.1 follows from Lemma 2.5.

Proof of Theorem 1.1. Applying Lemma 2.5 (i) to $(k_{i+1}, \ldots, k_n, k_1, \ldots, k_i)$, and summing up the result for $i = 1, \ldots, n$, we get Theorem 1.1 (i).

By using Lemma 2.5 (ii) and the same argument as above, we can prove Theorem 1.1 (ii). □

Remark 2.6. Differentiating both sides of the identities (i) and (ii) in Theorem 1.1, we can get relations among $Z(\mathbf{k}; \alpha)$ and $Z^\star(\mathbf{k}; \alpha)$, respectively. Further, taking $\alpha = 1$ in the results, we can get relations among MZVs and MZSVs, respectively. Clearly these relations among MZVs and MZSVs contain the cyclic sum formulas for MZVs and MZSVs, respectively.

Remark 2.7. This remark was added after the author submitted the first revised version of the present paper on 24 April 2009. Krattenthaler and Rivoal's hypergeometric identity [21, Proposition 1 (ii)], which is a non-terminating version of a limiting case of a basic hypergeometric identity of G.E. Andrews, contains the identity

$$2\zeta(2n+1; \alpha) = Z^\star(1, \underbrace{2, \ldots, 2}_{n}; \alpha) \tag{3}$$

for any $n \in \mathbb{Z}_{\geqslant 1}$ and all $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. (In the case $\alpha = 1$ in (3), see, e.g., [37].) Indeed, taking $a = 2\alpha$, $b_i = \alpha$ $(i = 1, \ldots, s)$, $c_j = \alpha$ $(j = 0, 1, \ldots, s-1)$ and $c_s = 1$, where $s \in \mathbb{Z}_{\geqslant 2}$ and $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < s-1$, in Proposition 1 (ii) in [21], we can get (3). The condition $0 < \operatorname{Re}\alpha < s-1$ can be changed into $\operatorname{Re}\alpha > 0$, because both sides of (3) are holomorphic in $\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha > 0\}$ as functions of α. We also note that the identity (3) is a special case of Theorem 1.1 (ii) (cf. [29, Examples (b)]).

As proved by M.E. Hoffman and Y. Ohno in [15] (see also [25]), and Y. Ohno and N. Wakabayashi in [29], the sum formulas for MZVs and MZSVs [13,14] follow from the cyclic sum formulas for MZVs and MZSVs, respectively. By using the same arguments, the following sum formulas for $Z(\mathbf{k}; \alpha)$ and $Z^\star(\mathbf{k}; \alpha)$ also follow from Theorem 1.1.

Corollary 2.8. *For any integers k, n with $0 < n < k$ and all complex numbers α with positive real part, the following identities hold*:

(i) [16, Proposition 1]

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_l \in \mathbb{Z}_{\geqslant 1}, k_n \geqslant 2}} Z(k_1, \ldots, k_n; \alpha) = \zeta(k; \alpha);$$

(ii)

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_l \in \mathbb{Z}_{\geqslant 1}, k_n \geqslant 2}} Z^\star(k_1, \ldots, k_n; \alpha) = \binom{k-1}{n-1} \zeta(k; \alpha).$$

In order to prove Corollary 2.8, we use the same argument as in [25, p. 4]. (The contents of [25, p. 4] can be also found in [26, pp. 138–139].) We need the following lemma.

Lemma 2.9 *(See Y. Ohno [25, Lemma 2], [26, Lemma 1]). Let* $\{a(k_1,\dots,k_n)\}$ *be a sequence. Then the identity*

$$\sum_{\substack{k_1+\cdots+k_n=k\\ k_l\in\mathbb{Z}_{\geqslant 1}}}\sum_{j=0}^{k_n-2} a(j+1,k_1,\dots,k_{n-1},k_n-j)=\sum_{\substack{k_1+\cdots+k_{n+1}=k\\ k_l\in\mathbb{Z}_{\geqslant 1}}} a(k_1,\dots,k_n,k_{n+1}+1) \tag{4}$$

holds for any integers k and n with $0<n<k$*, where the empty sum is interpreted as* 0*.*

Proof. In [25,26], Y. Ohno proved (4) by using a combinatorial argument. In this proof, we prove (4) by direct calculation. The left-hand side of (4) can be calculated as follows:

$$\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_n=k\\ k_l\in\mathbb{Z}_{\geqslant 1}}}\sum_{j=0}^{k_n-2} a(j+1,k_1,\dots,k_{n-1},k_n-j)\\
&=\sum_{k_n=1}^{k-n+1}\sum_{\substack{k_1+\cdots+k_{n-1}=k-k_n\\ k_l\in\mathbb{Z}_{\geqslant 1}}}\sum_{j=0}^{k_n-2} a(j+1,k_1,\dots,k_{n-1},k_n-j)\\
&=\sum_{k_n=1}^{k-n}\sum_{j=1}^{k_n}\sum_{\substack{k_1+\cdots+k_{n-1}=k-k_n-1\\ k_l\in\mathbb{Z}_{\geqslant 1}}} a(j,k_1,\dots,k_{n-1},k_n-j+2)\\
&=\sum_{j=1}^{k-n}\sum_{k_n=j}^{k-n}\sum_{\substack{k_1+\cdots+k_{n-1}=k-k_n-1\\ k_l\in\mathbb{Z}_{\geqslant 1}}} a(j,k_1,\dots,k_{n-1},k_n-j+2)\\
&=\sum_{j=1}^{k-n}\sum_{k_n=1}^{k-j-n+1}\sum_{\substack{k_1+\cdots+k_{n-1}=k-j-k_n\\ k_l\in\mathbb{Z}_{\geqslant 1}}} a(j,k_1,\dots,k_{n-1},k_n+1)\\
&=\sum_{\substack{j+k_1+\cdots+k_n=k\\ j,k_l\in\mathbb{Z}_{\geqslant 1}}} a(j,k_1,\dots,k_{n-1},k_n+1).
\end{aligned}$$

This completes the proof. □

Now we prove Corollary 2.8.

Proof of Corollary 2.8. Summing up both sides of the identity (i) in Theorem 1.1 for all $k_1,\dots,k_n\in\mathbb{Z}_{\geqslant 1}$ with $k_1+\cdots+k_n=k$, where $k,n\in\mathbb{Z}$ with $0<n<k$, we get

$$\sum_{i=1}^{n}\sum_{\substack{k_1+\cdots+k_n=k\\ k_l\in\mathbb{Z}_{\geqslant 1}}}\sum_{j=0}^{k_i-2} Z(j+1,k_{i+1},\dots,k_n,k_1,\dots,k_{i-1},k_i-j;\alpha)$$

$$= n \sum_{\substack{k_1+\cdots+k_n=k \\ k_l \in \mathbb{Z}_{\geqslant 1}}} Z(k_1, \ldots, k_{n-1}, k_n+1; \alpha).$$

Further, applying Lemma 2.9 to the inner sums on the left-hand side of the above identity, we get

$$n \sum_{\substack{k_1+\cdots+k_{n+1}=k \\ k_l \in \mathbb{Z}_{\geqslant 1}}} Z(k_1, \ldots, k_n, k_{n+1}+1; \alpha) = n \sum_{\substack{k_1+\cdots+k_n=k \\ k_l \in \mathbb{Z}_{\geqslant 1}}} Z(k_1, \ldots, k_{n-1}, k_n+1; \alpha).$$

Therefore, by induction on n, we get Corollary 2.8 (i).

Similarly, summing up both sides of the identity (ii) in Theorem 1.1 for all $k_1, \ldots, k_n \in \mathbb{Z}_{\geqslant 1}$ with $k_1 + \cdots + k_n = k$, where $k, n \in \mathbb{Z}$ with $0 < n < k$, and applying Lemma 2.9 to the result, we get

$$n \sum_{\substack{k_1+\cdots+k_{n+1}=k \\ k_l \in \mathbb{Z}_{\geqslant 1}}} Z^{\star}(k_1, \ldots, k_n, k_{n+1}+1; \alpha) = k \binom{k-1}{n-1} \zeta(k+1; \alpha).$$

Therefore, dividing both sides of the above identity by n, we get Corollary 2.8 (ii). □

Remark 2.10. In [16], the author proved a slightly general sum formula than Corollary 2.8 (i) (see also [17, Remark 2.5]) by using Ochiai's method of proving the sum formula for MZVs ([23]; though Ochiai's proof of the sum formula for MZVs is unpublished, it can be found in [3, pp. 17–20] and [19, pp. 60–61] (see also [17])). By using Hoffman's argument [14, p. 283], we can derive Corollary 2.8 (ii) from Corollary 2.8 (i) and vice versa. We also note that Corollary 2.8 (i) is a special case of Theorem 1.1 in [17].

Remark 2.11. In the study of MZVs, determining the linear independence of MZVs over $\mathbb{Q}$ is one of the main problems (see, e.g., [8, Section 2], [36]). For $Z(\mathbf{k}; \alpha)$, we can consider a problem similar to the above, i.e., determining the linear and the algebraic independence of the functions $Z(\mathbf{k}; \alpha)$ over the rational function field $\mathrm{K}(\alpha)$, where K is a field such that $\mathbb{Q} \subset \mathrm{K} \subset \mathbb{C}$. This problem is also inspired by the problem considered in [32, Section 3]. Now we give a remark related to the above problem for $Z(\mathbf{k}; \alpha)$. The following method of proving the linear independence of the functions $Z(\mathbf{k}; \alpha)$ is inspired by the method used in [33]. As an immediate consequence of the definition of $Z(\mathbf{k}; \alpha)$, we get the asymptotic relation

$$Z(k_1, \ldots, k_n; \alpha) \sim \begin{cases} \alpha^{-k_1} & \text{if } n = 1, k_1 \geqslant 2, \\ \zeta(k_2, \ldots, k_{n-1}, k_n - 1)\alpha^{-k_1-1} & \text{if } n \geqslant 2, k_n \geqslant 3 \end{cases}$$

as $\alpha \to 0$, $\operatorname{Re}\alpha > 0$, where the notation $f(x) \sim g(x)$ as $x \to a$ means that $f(x)/g(x) \to 1$ as $x \to a$. Using this asymptotic relation and the evaluations $\zeta(\underbrace{2n, \ldots, 2n}_{m}) \in \mathbb{Q}\pi^{2mn}$ ([3,4], [18, Corollary 2]) and $\zeta(\underbrace{1, 3, \ldots, 1, 3}_{2m}) \in \mathbb{Q}\pi^{4m}$ ([5,6]) for any $m, n \in \mathbb{Z}_{\geqslant 1}$, we can prove some results on the linear independence of the functions $Z(\mathbf{k}; \alpha)$ over the rational function field $\overline{\mathbb{Q}}(\alpha)$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. For example, by using the above asymptotic relation, L. Euler's result: $\zeta(2m) \in \mathbb{Q}\pi^{2m}$ for any $m \in \mathbb{Z}_{\geqslant 1}$, and F. Lindemann's result: $\pi \notin \overline{\mathbb{Q}}$, it is easy to prove the linear independence of the functions

$$\zeta(k; \alpha), \qquad Z(k-2n-1, 2n+1; \alpha),$$

$n = 1, \ldots, [(k-2)/2]$, over $\overline{\mathbb{Q}}(\alpha)$ for any $k \in \mathbb{Z}_{\geqslant 4}$, where $[x]$ denotes the greatest integer not greater than x. As an immediate consequence of the above example, we get the estimate

$$\dim \mathcal{Z}_{k,\overline{\mathbb{Q}}(\alpha)} \geqslant \left[\frac{k}{2}\right]$$

for any $k \in \mathbb{Z}_{\geqslant 2}$, where $\mathcal{Z}_{k,\overline{\mathbb{Q}}(\alpha)}$ denotes the vector space generated by all functions $Z(\mathbf{k};\alpha)$ with the weight $k \in \mathbb{Z}_{\geqslant 2}$ over the rational function field $\overline{\mathbb{Q}}(\alpha)$. Using this estimate and known relations among $Z(\mathbf{k};\alpha)$ (see [17]), we can estimate $\dim \mathcal{Z}_{k,\overline{\mathbb{Q}}(\alpha)}$ from above and below. For example, we get $\dim \mathcal{Z}_{4,\overline{\mathbb{Q}}(\alpha)} = 2$ and $2 \leqslant \dim \mathcal{Z}_{5,\overline{\mathbb{Q}}(\alpha)} \leqslant 3$. For the estimation of the dimension of the $\mathbb{Q}$-vector space generated by MZVs, see, e.g., [12,18,35].

Acknowledgments

The author is very grateful to the anonymous referee for his or her valuable comments on earlier versions of the present paper, which improved the presentation of the papers, providing a simple alternative proof of Lemma 2.2 and pointing out many mistakes.

The present research was done when the author was supported by a scholarship of Nagoya University. The author is grateful for this support.

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